

NON-UNIQUENESS OF WEAK SOLUTIONS TO HYPERVISCOUS NAVIER-STOKES EQUATIONS - ON SHARPNESS OF J.-L. LIONS EXPONENT

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ABSTRACT. Using the convex integration technique for the three-dimensional Navier-Stokes equations introduced by T. Buckmaster and V. Vicol, it is shown the existence of non-unique weak solutions for the 3D Navier-Stokes equations with fractional hyperviscosity $(-\Delta)^\theta$, whenever the exponent θ is less than J.-L. Lions' exponent $5/4$, i.e., when $\theta < 5/4$.

1. INTRODUCTION

In this paper we consider the question of non-uniqueness of weak solutions to the 3D Navier-Stokes equations with fractional viscosity (FVNSE) on \mathbb{T}^3

$$\begin{cases} \partial_t v + \nabla \cdot (v \otimes v) + \nabla p + \nu(-\Delta)^\theta v = 0, \\ \nabla \cdot v = 0, \end{cases} \quad (1)$$

where $\theta \in \mathbb{R}$ is a fixed constant, and for $u \in C^\infty(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} u(x) dx = 0$, the fractional Laplacian is defined via the Fourier transform as

$$\mathcal{F}((-\Delta)^\theta u)(\xi) = |\xi|^{2\theta} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^3.$$

Definition (weak solutions). *A vector field $v \in C_{weak}^0(\mathbb{R}; L^2(\mathbb{T}^3))$ is called a weak solution to the FVNSE if it solves (1) in the sense of distribution.*

When $\theta = 1$, FVNSE (1) is the standard Navier-Stokes equations. J.-L. Lions first considered FVNSE (1) in [20], and showed the existence and uniqueness of weak solutions to the initial value problem, which also satisfied the energy equality, for $\theta \in [5/4, \infty)$ in [21]. Moreover, an analogue of the Caffarelli-Kohn-Nirenberg [6] result was established in [18] for the FVNSE system (1), showing that the Hausdorff dimension of the singular set, in space and time, is bounded by $5 - 4\theta$ for $\theta \in (1, 5/4)$. The existence, uniqueness, regularity and stability of solutions to the FVNSE have been studied in [17, 26, 28, 29] and references therein. Very recently, using the method of convex integration introduced in [12], Colombo, De Lellis and

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De Rosa in [8] showed the non-uniqueness of Leray weak solutions to FVNSE (1) for $\theta \in (0, 1/5)$ and for $\theta \in (0, 1/3)$ in [13].

In the recent breakthrough work [5], Buckmaster and Vicol obtained non-uniqueness of weak solutions to the three-dimensional Navier-Stokes equations. They developed a new convex integration scheme in Sobolev spaces using intermittent Beltrami flows which combined concentrations and oscillations. Later, the idea of using intermittent flows was used to study non-uniqueness for transport equations in [23–25] employing scaled Mikado waves, and for stationary Navier-Stokes equations in [7, 22] employing viscous eddies.

The schemes in [5, 24] are based on the convex integration framework in Hölder spaces for the Euler equations, introduced by De Lellis and Székelyhidi in [12], subsequently refined in [2, 3, 10, 15], and culminated in the proof of the second half of the Onsager conjecture by Isett in [16]; also see [4] for a shorter proof. For the first half of the Onsager conjecture, see, e.g., [1, 9], and the references therein.

The main contribution of this note is to show that the results in Buckmaster-Vicol's paper hold for FVNSE (1) for $\theta < 5/4$:

Theorem 1. *Assume that $\theta \in [1, 5/4)$. Suppose u is a smooth divergence-free vector field, define on $\mathbb{R}_+ \times \mathbb{T}^3$, with compact support in time and satisfies the condition*

$$\int_{\mathbb{T}^3} u(t, x) dx \equiv 0.$$

Then for any given $\varepsilon_0 > 0$, there exists a weak solution v to the FVNSE (1), with compact support in time, satisfying

$$\|v - u\|_{L_t^\infty W_x^{2\theta-1,1}} < \varepsilon_0.$$

As a consequence there are infinitely many weak solutions of the FVNSE (1) which are compactly supported in time; in particular, there are infinitely many weak solutions with initial values zero.

Remark 1. *In the above theorem we assume that $\theta \in [1, 5/4)$. However, using the constructions in [5] with a slightly different choice of parameters, one can actually show that Theorem 1.2 and Theorem 1.3 in [5] hold for the 3D FVNSE, i.e., there exist non-unique weak solutions $v \in C_t^0 W_x^{\beta,2}$, with a different $\beta > 0$, depending on θ . However, in this paper we choose to prove a weaker result, Theorem 1, in order to simplify the presentation while retaining the main idea.*

Remark 2. *For the case $\theta \in (-\infty, 1)$, the same construction also yields weak solutions $v \in C_t^0 L_x^2 \cap C_t^0 W_x^{1,1}$ with a suitable choice of parameters.*

We now make some comments on the analysis in this paper. Using the technique in [5], we adapt a convex integration scheme with intermittent Beltrami flows as the building blocks. The main difficulty in a convex integration scheme for (FVNSE), is the error induced by the frictional viscosity $\nu(-\Delta)^\theta v$, which is greater for a larger exponent θ . This error is controlled by making full use of the concentration effect of intermittent flows introduced in [5]. As it is shown in the crucial estimate (36), the error is controllable only for $\theta < 5/4$. Compared with [5], since our goal is to construct weak solutions $v \in C_t^0 L_{x,weak}^2 \cap L_t^\infty W_x^{2\theta-1,1}$, we adapt a slightly simpler cut-off function and prove only estimates that are sufficient for this purpose.

2. OUTLINE

2.1. **Iteration lemma.** Following [5], we consider the approximate system

$$\begin{cases} \partial_t v + \nabla \cdot (v \otimes v) + \nabla p + \nu(-\Delta)^\theta v = \nabla \cdot R, \\ \nabla \cdot v = 0, \end{cases} \quad (2)$$

where R is a symmetric 3×3 matrix.

Lemma 1 (Iteration Lemma for L^2 weak solutions). *Let $\theta \in (-\infty, 5/4)$. Assume (v_q, R_q) is a smooth solution to (2) with*

$$\|R_q\|_{L_t^\infty L_x^1} \leq \delta_{q+1}, \quad (3)$$

for some $\delta_{q+1} > 0$. Then for any given $\delta_{q+2} > 0$, there exists a smooth solution (v_{q+1}, R_{q+1}) of (2) with

$$\|R_{q+1}\|_{L_t^\infty L_x^1} \leq \delta_{q+2}, \quad (4)$$

$$\text{and } \text{supp}_t v_{q+1} \cup \text{supp}_t R_{q+1} \subset N_{\delta_{q+1}}(\text{supp}_t v_q \cup \text{supp}_t R_q). \quad (5)$$

Here for a given set $A \subset \mathbb{R}$, the δ -neighborhood of A is denoted by

$$N_\delta(A) = \{y \in \mathbb{R} : \exists y' \in A, |y - y'| < \delta\}.$$

Furthermore, the increment $w_{q+1} = v_{q+1} - v_q$ satisfies the estimates

$$\|w_{q+1}\|_{L_t^\infty L_x^2} \leq C \delta_{q+1}^{1/2}, \quad (6)$$

$$\|w_{q+1}\|_{L_t^\infty W_x^{2\theta-1,1}} \leq \delta_{q+2}, \quad (7)$$

where the positive constant C depends only on θ .

Proof of Theorem 1. Assume Lemma 1 is valid. Let $v_0 = u$. Then

$$\int_{\mathbb{T}^3} \partial_t v_0(t, x) dx = \frac{d}{dt} \int_{\mathbb{T}^3} v_0(t, x) dx \equiv 0.$$

Let

$$R_0 = \mathcal{R}(\partial_t v_0 + \nu(-\Delta)^\theta v_0) + v_0 \otimes v_0 + p_0 I, \quad p_0 = -\frac{1}{3}|v_0|^2,$$

where \mathcal{R} is the symmetric anti-divergence operator established in Lemma 5, below. Clearly (v_0, R_0) solves (2). Set

$$\delta_1 = \|R_0\|_{L_t^\infty L_x^1},$$

$$\delta_{q+1} = 2^{-q} \varepsilon_0, \quad \text{for } q \geq 1.$$

Apply Lemma 1 iteratively to obtain smooth solution (v_q, R_q) to (2). It follows from (6) that

$$\sum \|v_{q+1} - v_q\|_{L_t^\infty L_x^2} = \sum \|w_{q+1}\|_{L_t^\infty L_x^2} \leq C \sum \delta_{q+1}^{1/2} < \infty.$$

Thus v_q converge strongly to some $v \in C_t^0 L_x^2$. Since $\|R_{q+1}\|_{L_t^\infty L_x^1} \rightarrow 0$, as $q \rightarrow \infty$, v is a weak solution to the FVNSE (1). Estimate (7) leads to

$$\|v - v_0\|_{L_t^\infty W_x^{2\theta-1,1}} \leq \sum_{q=1}^{\infty} \|w_q\|_{L_t^\infty W_x^{2\theta-1,1}} \leq \sum_{q=1}^{\infty} \delta_{q+1} \leq \varepsilon_0.$$

Furthermore, it follows from (5) that

$$\text{supp}_t v \subset \cup_{q \geq 0} \text{supp}_t v_q \subset N_{\sum_{q \geq 0} \delta_{q+1}}(\text{supp}_t u) \subset N_{\delta_1 + \varepsilon_0}(\text{supp}_t u).$$

Now we show the existence of infinitely many weak solutions with initial values zero. Let $u(t, x) = \varphi(t) \sum_{|k| \leq N} a_k e^{ik \cdot x}$ with $a_k \neq 0, a_k \cdot k = 0, a_{-k} = a_k^*$ for all $|k| \leq N$, and $\varphi \in C_c^\infty(\mathbb{R}_+)$. Thus $\nabla \cdot u = 0$ satisfies the conditions of the theorem. Hence there exists a weak solution v to (1) close enough to u so that $v \neq 0$. \square

3. ITERATION SCHEME

3.1. Notations and Parameters. For a complex number $\zeta \in \mathbb{C}$, we denote by ζ^* its complex conjugate. Let us normalize the volume

$$|\mathbb{T}^3| = 1.$$

For smooth functions $u \in C^\infty(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} u(x) dx = 0$ and $s \in \mathbb{R}$, we define

$$\mathcal{F}(|\nabla|^s u)(\xi) = |\xi|^s \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^3.$$

For $M, N \in [0, +\infty]$, denote the Fourier projection of u by

$$\mathcal{F}(\mathbb{P}_{[M, N)} u) = \begin{cases} u(\xi), & M \leq |\xi| < N, \xi \in \mathbb{Z}^3, \\ 0, & \text{otherwise.} \end{cases}$$

We also denote $\mathbb{P}_{\leq k} = \mathbb{P}_{[0, k]}$ and $\mathbb{P}_{\geq k} = \mathbb{P}_{[k, +\infty)}$ for $k > 0$.

Following the notation in [5], we introduce here several parameters σ, r, λ , with

$$0 < \sigma < 1 < r < \lambda < \mu < \lambda^2, \quad \sigma r < 1, \quad (8)$$

where $\lambda = \lambda_{q+1} \in 5\mathbb{N}$ is the ‘frequency’ parameter; σ with $1/\sigma \in \mathbb{N}$ is a small parameter such that $\lambda\sigma \in \mathbb{N}$ parameterizes the spacing between frequencies; $r \in \mathbb{N}$ denotes the number of frequencies along edges of a cube; μ measures the amount of temporal oscillation.

Later σ, r, μ will be chosen to be suitable powers of λ_{q+1} . We also fix a constant $p > 1$ which will be chosen later to be close to 1. The constants implicitly in the notation ‘ \lesssim ’ may depend on p but are independent of the parameters σ, r, λ .

3.2. Intermittent Beltrami flows. We use intermittent Beltrami flows introduced in [5] as the building blocks. Recall some basic facts of Beltrami waves.

Proposition 1. ([5, Proposition 3.1]) *Given $\bar{\xi} \in \mathbb{S}^2 \cap \mathbb{Q}^3$, let $A_{\bar{\xi}} \in \mathbb{S}^2 \cap \mathbb{Q}^3$ be such that*

$$A_{\bar{\xi}} \cdot \bar{\xi} = 0, \quad |A_{\bar{\xi}}| = 1, \quad A_{-\bar{\xi}} = A_{\bar{\xi}}.$$

Let Λ be a given finite subset of \mathbb{S}^2 such that $-\Lambda = \Lambda$, and $\lambda \in \mathbb{Z}$ be such that $\lambda\Lambda \subset \mathbb{Z}^3$. Then for any choice of coefficients $a_{\bar{\xi}} \in \mathbb{C}$ with $a_{\bar{\xi}}^ = a_{-\bar{\xi}}$ the vector field*

$$W(x) = \sum_{\bar{\xi} \in \Lambda} a_{\bar{\xi}} B_{\bar{\xi}} e^{i\lambda \bar{\xi} \cdot x}, \quad \text{with } B_{\bar{\xi}} = \frac{1}{\sqrt{2}}(A_{\bar{\xi}} + i\bar{\xi} \times A_{\bar{\xi}}),$$

is real-valued, divergence-free and satisfies

$$\nabla \times W = \lambda W, \quad \nabla \cdot (W \otimes W) = \nabla \frac{|W|^2}{2}.$$

Furthermore,

$$\langle W \otimes W \rangle := \int_{\mathbb{T}^3} W \otimes W dx = \sum_{\bar{\xi} \in \Lambda} \frac{1}{2} |a_{(\bar{\xi})}|^2 (\text{Id} - \bar{\xi} \otimes \bar{\xi}).$$

Let $\Lambda, \Lambda^+, \Lambda^- \subset \mathbb{S}^2 \cap \mathbb{Q}^3$ be defined by

$$\begin{aligned} \Lambda^+ &= \left\{ \frac{1}{5}(3e_1 \pm 4e_2), \frac{1}{5}(3e_2 \pm 4e_3), \frac{1}{5}(3e_3 \pm 4e_1) \right\}, \\ \Lambda^- &= -\Lambda^+, \quad \Lambda = \Lambda^+ \cup \Lambda^-. \end{aligned}$$

Clearly we have

$$5\Lambda \in \mathbb{Z}^3, \quad \text{and} \quad \min_{\bar{\xi}', \bar{\xi} \in \Lambda, \bar{\xi}' + \bar{\xi} \neq 0} |\bar{\xi}' + \bar{\xi}| \geq \frac{1}{5}. \quad (9)$$

Also it is direct to check that

$$\frac{1}{8} \sum_{\bar{\xi} \in \Lambda} (\text{Id} - \bar{\xi} \otimes \bar{\xi}) = \text{Id}.$$

In fact, representations of this form exist for symmetric matrices close to the identity. We have the following simple variant of [5, Proposition 3.2].

Proposition 2. *Let $B_\varepsilon(\text{Id})$ denote the ball of symmetric matrices, centered at the identity, of radius ε . Then there exist a constant $\varepsilon_\gamma > 0$ and smooth positive functions $\gamma_{(\bar{\xi})} \in C^\infty(B_{\varepsilon_\gamma}(\text{Id}))$, such that*

- (1) $\gamma_{(\bar{\xi})} = \gamma_{(-\bar{\xi})}$;
- (2) *for each $R \in B_{\varepsilon_\gamma}(\text{Id})$ we have the identity*

$$R = \frac{1}{2} \sum_{\bar{\xi} \in \Lambda} \left(\gamma_{(\bar{\xi})}(R) \right)^2 (\text{Id} - \bar{\xi} \otimes \bar{\xi}).$$

Define the Dirichlet kernel

$$D_r(x) = \frac{1}{(2r+1)^{3/2}} \sum_{\xi \in \Omega_r} e^{i\xi \cdot x}, \quad \Omega_r = \{(j, k, l) : j, k, l \in \{-r, \dots, r\}\}.$$

It has the property that, for $1 < p \leq \infty$,

$$\|D_r\|_{L^p} \lesssim r^{3/2-3/p}, \quad \|D_r\|_{L^2} = (2\pi)^3.$$

Following [5], for $\bar{\xi} \in \Lambda^+$, define a directed and rescaled Dirichlet kernel by

$$\eta_{(\bar{\xi})}(t, x) = \eta_{\bar{\xi}, \lambda, \sigma, r, \mu}(t, x) = D_r(\lambda \sigma(\bar{\xi} \cdot x + \mu t, A_{\bar{\xi}} \cdot x, (\bar{\xi} \times A_{\bar{\xi}}) \cdot x)), \quad (10)$$

and for $\bar{\xi} \in \Lambda^-$, define

$$\eta_{(\bar{\xi})}(t, x) = \eta_{-(\bar{\xi})}(t, x).$$

Note the important identity

$$\frac{1}{\mu} \partial_t \eta_{(\bar{\xi})}(t, x) = \pm (\bar{\xi} \cdot \nabla) \eta_{(\bar{\xi})}(t, x), \quad \bar{\xi} \in \Lambda^\pm. \quad (11)$$

Since the map $x \mapsto \lambda \sigma(\bar{\xi} \cdot x + \mu t, A_{\bar{\xi}} \cdot x, (\bar{\xi} \times A_{\bar{\xi}}) \cdot x)$ is the composition of a rotation by a rational orthogonal matrix mapping $\{e_1, e_2, e_3\}$ to $\{\bar{\xi}, A_{\bar{\xi}}, \bar{\xi} \times A_{\bar{\xi}}\}$, a translation, and a rescaling by integers, for $1 < p \leq \infty$, we have

$$\oint_{\mathbb{T}^3} \eta_{(\bar{\xi})}(t, x)^2(t, x) dx = 1, \quad \|\eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p(\mathbb{T}^3)} \lesssim r^{3/2-3/p}.$$

Let $W_{(\bar{\xi})}$ be the Beltrami plane wave at frequency λ ,

$$W_{(\bar{\xi})} = W_{\bar{\xi}, \lambda}(x) = B_{\bar{\xi}} e^{i\lambda \bar{\xi} \cdot x}.$$

Define the intermittent Beltrami wave $\mathbb{W}_{(\bar{\xi})}$ as

$$\mathbb{W}_{(\bar{\xi})}(t, x) := \mathbb{W}_{\bar{\xi}, \lambda, \sigma, r, \mu}^-(t, x) = \eta_{(\bar{\xi})}(t, x) W_{(\bar{\xi})}(x). \quad (12)$$

It follows from the definitions and (9) that

$$\mathbb{P}_{[\frac{\lambda}{2}, 2\lambda)} \mathbb{W}_{(\bar{\xi})} = \mathbb{W}_{(\bar{\xi})}, \quad (13)$$

$$\mathbb{P}_{[\frac{\lambda}{5}, 4\lambda)}(\mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(\bar{\xi}')}) = \mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(\bar{\xi}')}, \quad \bar{\xi}' \neq -\bar{\xi}. \quad (14)$$

The following properties are immediate from the definitions.

Proposition 3. (*[5, Proposition 3.4]*) Let $a_{\bar{\xi}} \in \mathbb{C}$ be constants with $a_{\bar{\xi}}^* = a_{-\bar{\xi}}$. Let

$$W(x) = \sum_{\bar{\xi} \in \Lambda} a_{\bar{\xi}} \mathbb{W}_{(\bar{\xi})}(x).$$

Then $W(x)$ is real valued. Moreover, for each $R \in B_{\varepsilon_\gamma}(\text{Id})$ we have

$$\sum_{\bar{\xi} \in \Lambda} \left(\gamma_{(\bar{\xi})}(R) \right)^2 \int_{\mathbb{T}^3} \mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(-\bar{\xi})} = \sum_{\bar{\xi} \in \Lambda} \left(\gamma_{(\bar{\xi})}(R) \right)^2 B_{\bar{\xi}} \otimes B_{-\bar{\xi}} = R.$$

Proposition 4. (*[5, Proposition 3.5]*) For any $1 < p \leq \infty, N \geq 0, K \geq 0$:

$$\|\nabla^N \partial_t^K \mathbb{W}_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \lesssim \lambda^N (\lambda \sigma r \mu)^K r^{3/2-3/p}, \quad (15)$$

$$\|\nabla^N \partial_t^K \eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \lesssim (\lambda \sigma r)^N (\lambda \sigma r \mu)^K r^{3/2-3/p}. \quad (16)$$

3.3. Perturbations. Let $\psi(t)$ be a smooth cut-off function such that

$$\psi(t) = 1 \text{ on } \text{supp}_t R_q, \quad \text{supp } \psi(t) \subset N_{\delta_{q+1}}(\text{supp}_t R_q), \quad |\psi'(t)| \leq 2\delta_{q+1}^{-1}. \quad (17)$$

Take a smooth increasing function χ such that

$$\chi(s) = \begin{cases} 1, & 0 \leq s < 1 \\ s, & s \geq 1 \end{cases},$$

and set

$$\rho(t, x) = \varepsilon_\gamma^{-1} \delta_{q+1} \chi(\delta_{q+1}^{-1} |R_q(t, x)|) \psi^2(t).$$

where ε_γ is the constant in Proposition 2. Then clearly

$$\text{supp}_t \rho \subset N_{\delta_{q+1}}(\text{supp}_t R_q). \quad (18)$$

It follows from the above definition that

$$|R_q|/\rho = \varepsilon_\gamma \frac{|R_q|}{\delta_{q+1} \chi(\delta_{q+1}^{-1} |R_q(t, x)|) \psi^2} \leq \varepsilon_\gamma \implies \text{Id} - R_q/\rho \in B_{\varepsilon_\gamma}(\text{Id}) \text{ on } \text{supp } R_q.$$

Therefore, the amplitude functions

$$a_{(\bar{\xi})}(t, x) := \rho^{1/2}(t, x) \gamma_{(\bar{\xi})}(\text{Id} - \rho(t, x)^{-1} R_q(t, x))$$

are well-defined and smooth. Define the velocity perturbation to be $w = w_{q+1}$:

$$\begin{aligned} w &= w^{(p)} + w^{(c)} + w^{(t)}, \\ w^{(p)} &= \sum_{\bar{\xi} \in \Lambda} a_{(\bar{\xi})} \mathbb{W}_{(\bar{\xi})} = \sum_{\bar{\xi} \in \Lambda} a_{(\bar{\xi})}(t, x) \eta_{(\bar{\xi})}(t, x) B_{\bar{\xi}} e^{i\lambda \bar{\xi} \cdot x}, \\ w^{(c)} &= \frac{1}{\lambda_{q+1}} \sum_{\bar{\xi} \in \Lambda} \nabla \left(a_{(\bar{\xi})} \eta_{(\bar{\xi})} \right) \times W_{(\bar{\xi})}, \\ w^{(t)} &= \frac{1}{\mu} \sum_{\bar{\xi} \in \Lambda^+} \mathbb{P}_{LH} \mathbb{P}_{\neq 0} \left(a_{(\bar{\xi})}^2 \eta_{(\bar{\xi})}^2 \bar{\xi} \right), \end{aligned}$$

where $\mathbb{P}_{LH} = \text{Id} - \nabla \Delta^{-1} \text{div}$ is the Leray-Helmholtz projection into divergence-free vector field, and $\mathbb{P}_{\neq 0} f = f - \int_{\mathbb{T}^3} f dx$. It is well-known that \mathbb{P}_{LH} is bounded on L^p , $1 < p < \infty$ (see, e.g., [14]). It follows from Proposition 3 that

$$\sum_{\bar{\xi} \in \Lambda} a_{(\bar{\xi})}^2 \int_{\mathbb{T}^3} \mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(-\bar{\xi})} dx = \rho \text{Id} - R_q. \quad (19)$$

3.4. Estimates for perturbations.

Lemma 2. *The following bounds hold:*

$$\|\rho\|_{L_t^\infty L_x^1} \leq C \delta_{q+1}, \quad (20)$$

$$\|\rho^{-1}\|_{C^0(\text{supp } R_q)} \lesssim \delta_{q+1}^{-1}, \quad (21)$$

$$\|\rho\|_{C_{t,x}^N} \leq C(\delta_{q+1}, \|R_q\|_{C^N}), \quad (22)$$

$$\|a_{(\bar{\xi})}\|_{L_t^\infty L_x^2} \lesssim \|\rho\|_{L_t^\infty L_x^1}^{1/2} \lesssim \delta_{q+1}^{1/2}, \quad (23)$$

$$\|a_{(\bar{\xi})}\|_{C_{t,x}^N} \leq C(\delta_{q+1}, \|R_q\|_{C^N}). \quad (24)$$

Proof. It follows from (3) that

$$\begin{aligned} \|\rho(t, \cdot)\|_{L_x^1} &= \int_{|R_q| \leq \delta_{q+1}} \rho + \int_{|R_q| > \delta_{q+1}} \rho \lesssim \delta_{q+1} + \int_{|R_q| > \delta_{q+1}} |R_q| \\ &\leq C \delta_{q+1}. \end{aligned}$$

It is direct to verify (21) and (23), while (22) and (24) follow from (17) and (21). \square

Now we can estimate the time support of w_{q+1} :

$$\text{supp}_t w_{q+1} \subset \text{supp}_t \rho \subset \text{supp } \psi \subset N_{\delta_{q+1}}(\text{supp}_t R_q). \quad (25)$$

We need the following Lemma, which is a variant of [5, Lemma 3.6].

Lemma 3. ([24, Lemma 2.1]) *Let $f, g \in C^\infty(\mathbb{T}^3)$, and g is $(\mathbb{T}/N)^3$ periodic, $N \in \mathbb{N}$. Then for $1 \leq p \leq \infty$,*

$$\|fg\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^p} + C_p N^{-1/p} \|f\|_{C^1} \|g\|_{L^p}.$$

Let us denote

$$\mathcal{C}_N = C(\sup_{\bar{\xi} \in \Lambda} \|a_{(\bar{\xi})}\|_{C_{t,x}^N}) \quad (26)$$

to be some polynomials depending on $\sup_{\bar{\xi} \in \Lambda} \|a_{(\bar{\xi})}\|_{C_{t,x}^N}$.

Lemma 4. *Suppose the parameters satisfy (8) and*

$$r^{3/2} \leq \mu. \quad (27)$$

Then the following estimates for the perturbations hold:

$$\|w_{q+1}^{(p)}\|_{L_t^\infty L_x^2} \lesssim \delta_{q+1}^{1/2} + (\lambda_{q+1}\sigma)^{-1/2}\mathcal{C}_1, \quad (28)$$

$$\|w_{q+1}\|_{L_t^\infty L_x^p} \lesssim r^{3/2-3/p}\mathcal{C}_1, \quad (29)$$

$$\|w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} + \|w_{q+1}^{(t)}\|_{L_t^\infty L_x^p} \lesssim (\sigma r + \mu^{-1}r^{3/2})r^{3/2-3/p}\mathcal{C}_1, \quad (30)$$

$$\|\partial_t w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} + \|\partial_t w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} \lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2, \quad (31)$$

$$\| |\nabla|^N w_{q+1} \|_{L_t^\infty L_x^p} \lesssim r^{3/2-3/p}\lambda_{q+1}^N \mathcal{C}_{N+1}, \quad (32)$$

for $1 < p < \infty$, $N \geq 1$.

Proof. Since $\mathbb{W}_{(\bar{\xi})}$ is $(\mathbb{T}/\lambda\sigma)^3$ periodic, it follows from (15), (23), and Lemma 3 that

$$\begin{aligned} \|w_{q+1}^{(p)}\|_{L_t^\infty L_x^2} &\lesssim \sum_{\bar{\xi} \in \Lambda} (\|a_{(\bar{\xi})}\|_{L_t^\infty L_x^2} + (\lambda_{q+1}\sigma)^{-1/2}\|a_{(\bar{\xi})}\|_{C^1}) \|\mathbb{W}_{(\bar{\xi})}\|_{L_t^\infty L_x^2} \\ &\lesssim \delta_{q+1}^{1/2} + (\lambda_{q+1}\sigma)^{-1/2}\mathcal{C}_1. \end{aligned}$$

In view of (8), (15) and (16) yield that

$$\begin{aligned} \|w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} &\lesssim \sum_{\bar{\xi} \in \Lambda} \|a_{(\bar{\xi})}\|_{C^0} \|\mathbb{W}_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \lesssim r^{3/2-3/p}\mathcal{C}_0, \\ \|w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1} \sum_{\bar{\xi} \in \Lambda} \left(\|\eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} + \|\nabla \eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \right) \|a_{(\bar{\xi})}\|_{C^1} \|\mathbb{W}_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \\ &\lesssim (\sigma r) r^{3/2-3/p}\mathcal{C}_1, \\ \|w_{q+1}^{(t)}\|_{L_t^\infty L_x^p} &\lesssim \mu^{-1} \sum_{\bar{\xi} \in \Lambda^+} \|a_{(\bar{\xi})}^2\|_{C^0} \|\eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p}^2 \lesssim \mu^{-1} \sum_{\bar{\xi} \in \Lambda^+} \|a_{(\bar{\xi})}^2\|_{C^0} \|\eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p}^2 \\ &\lesssim \mu^{-1} r^{3-3/p}\mathcal{C}_0, \end{aligned}$$

where the boundedness of \mathbb{P}_{LH} and $\mathbb{P}_{\neq 0}$ on L^p , for $1 < p < \infty$, is used in the first inequality of the estimate for $\|w_{q+1}^{(t)}\|_{L_t^\infty L_x^p}$. In the same way, we can estimate

$$\begin{aligned} \|\partial_t w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} &\lesssim \sum_{\bar{\xi} \in \Lambda} \|\partial_t a_{(\bar{\xi})}\|_{C^0} \|\mathbb{W}_{(\bar{\xi})}\|_{L_t^\infty L_x^p} + \|a_{(\bar{\xi})}\|_{C^0} \|\partial_t \mathbb{W}_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \\ &\lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_1, \\ \|\partial_t w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1} \sum_{\bar{\xi} \in \Lambda} \|a_{(\bar{\xi})}\|_{C_{t,x}^2} \left(\|\eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} + \|\nabla \eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} + \|\partial_t \eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \right. \\ &\quad \left. + \|\partial_t \nabla \eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \right) \lesssim \sigma r \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2 \lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2. \end{aligned}$$

For $N \geq 1$, Using (15) and (16), we obtain that

$$\begin{aligned}
\|\nabla^N w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} &\lesssim \sum_{\bar{\xi} \in \Lambda} \sum_{k=0}^N \|\nabla^k a_{(\bar{\xi})}\|_{C^0} \|\nabla^{N-k} \mathbb{W}_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p} \mathcal{C}_N, \\
\|\nabla^N w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1} \sum_{\bar{\xi} \in \Lambda} \sum_{m=0}^N \sum_{k=0}^m \lambda_{q+1}^{N-m} \|\nabla^{k+1} a_{(\bar{\xi})}\|_{C^0} \|\nabla^{m-k} \eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \\
&\quad + \lambda_{q+1}^{-1} \sum_{\bar{\xi} \in \Lambda} \sum_{m=0}^N \sum_{k=0}^m \lambda_{q+1}^{N-m} \|\nabla^k a_{(\bar{\xi})}\|_{C^0} \|\nabla^{m-k+1} \eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p} \mathcal{C}_{N+1}, \\
\|\nabla^N w_{q+1}^{(t)}\|_{L_t^\infty L_x^p} &\lesssim \mu^{-1} \sum_{\bar{\xi} \in \Lambda} \sum_{m=0}^N \|\nabla^{N-m} (a_{(\bar{\xi})}^2)\|_{C^0} \sum_{k=0}^m \|\nabla^k \eta_{(\bar{\xi})}\|_{L_t^\infty L_x^{2p}} \|\nabla^{m-k} \eta_{(\bar{\xi})}\|_{L_t^\infty L_x^{2p}} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p} \frac{(\sigma r)^N r^{3/2}}{\mu} \mathcal{C}_N \lesssim \lambda_{q+1}^N r^{3/2-3/p} \mathcal{C}_N,
\end{aligned}$$

where we use (8) and (27). \square

3.5. Estimates for the stress. Let us recall the following operator in [12].

Lemma 5 (symmetric anti-divergence). *There exists a linear operator \mathcal{R} , of order -1 , mapping vector fields to symmetric matrices such that*

$$\nabla \cdot \mathcal{R}(u) = u - \oint_{\mathbb{T}^3} u, \quad (33)$$

with standard Calderon-Zygmund estimates, for $1 < p < \infty$,

$$\|\mathcal{R}\|_{L^p \rightarrow W^{1,p}} \lesssim 1, \quad \|\mathcal{R}\|_{C^0 \rightarrow C^0} \lesssim 1, \quad \|\mathcal{R} \mathbb{P}_{\neq 0} u\|_{L^p} \lesssim \| |\nabla|^{-1} \mathbb{P}_{\neq 0} u \|_{L^p}. \quad (34)$$

Proof. Suppose $u \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ is a smooth vector field. Define

$$\mathcal{R}(u) = \frac{1}{4} (\nabla \mathbb{P}_{LH} v + (\nabla \mathbb{P}_{LH} v)^T) + \frac{3}{4} (\nabla v + (\nabla v)^T) - \frac{1}{2} (\nabla \cdot v) \text{Id}$$

where $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ is the unique solution to $\Delta v = u - \oint_{\mathbb{T}^3} u$ with $\oint_{\mathbb{T}^3} v = 0$.

It is direct to verify that $\mathcal{R}(u)$ is a symmetric matrix field depending linearly on u and satisfies (33). Note that \mathcal{R} is a constant coefficient elliptic operator of order -1 . We refer to [14] for the Calderon-Zygmund estimates $\|\mathcal{R}\|_{L^p \rightarrow W^{1,p}} \lesssim 1$ and $\|\mathcal{R} \mathbb{P}_{\neq 0} u\|_{L^p} \lesssim \| |\nabla|^{-1} \mathbb{P}_{\neq 0} u \|_{L^p}$. Combining these with Sobolev embeddings, we have $\|\mathcal{R} u\|_{C^\alpha} \lesssim \|\mathcal{R} u\|_{W^{1,4}} \lesssim \|u\|_{L^4} \lesssim \|u\|_{C^0}$, with $\alpha = 1/4$. \square

We have the following variant of [5, Lemma B.1] in [5].

Lemma 6. *Let $a \in C^2(\mathbb{T}^3)$. For $1 < p < \infty$, and any smooth function $f \in L^p(\mathbb{T}^3)$, we have*

$$\| |\nabla|^{-1} \mathbb{P}_{\neq 0} (a \mathbb{P}_{\geq k} f) \|_{L^p(\mathbb{T}^3)} \lesssim k^{-1} \|\nabla^2 a\|_{L^\infty(\mathbb{T}^3)} \|f\|_{L^p(\mathbb{T}^3)}. \quad (35)$$

Proof of Lemma 6. We follow the proof in [5]. Note that

$$| \nabla |^{-1} \mathbb{P}_{\neq 0} (a \mathbb{P}_{\geq k} f) = | \nabla |^{-1} \mathbb{P}_{\geq k/2} (\mathbb{P}_{\leq k/2} a \mathbb{P}_{\geq k} f) + | \nabla |^{-1} \mathbb{P}_{\neq 0} (\mathbb{P}_{\geq k/2} a \mathbb{P}_{\geq k} f).$$

As direct consequences of the Littlewood-Paley decomposition and Schauder estimates we have the bounds for $1 < p < \infty$ (see, for example, [14])

$$\|\mathbb{P}_{\leq k/2}\|_{L^p \rightarrow L^p} \lesssim 1, \quad \||\nabla|^{-1}\mathbb{P}_{\geq k/2}\|_{L^p \rightarrow L^p} \lesssim k^{-1}, \quad \||\nabla|^{-1}\mathbb{P}_{\neq 0}\|_{L^p \rightarrow L^p} \lesssim 1.$$

Combining these bounds with Hölder's inequality and the embedding $W^{1,4}(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$, we obtain

$$\begin{aligned} & \||\nabla|^{-1}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geq k}f)\|_{L^p} \lesssim k^{-1}\|\mathbb{P}_{\leq k/2}a\mathbb{P}_{\geq k}f\|_{L^p} + \|\mathbb{P}_{\geq k/2}a\mathbb{P}_{\geq k}f\|_{L^p} \\ & \lesssim k^{-1}(\|\mathbb{P}_{\leq k/2}a\|_{L^\infty} + k\|\mathbb{P}_{\geq k/2}a\|_{L^\infty})\|f\|_{L^p} \lesssim k^{-1}(\|\nabla\mathbb{P}_{\leq k/2}a\|_{L^4} + k\|\nabla\mathbb{P}_{\geq k/2}a\|_{L^4})\|f\|_{L^p} \\ & \lesssim k^{-1}(\|\mathbb{P}_{\leq k/2}\nabla a\|_{L^4} + k\||\nabla|^{-1}\mathbb{P}_{\geq k/2}|\nabla|\nabla\mathbb{P}_{\geq k/2}a\|_{L^4})\|f\|_{L^p} \\ & \lesssim k^{-1}(\|\nabla a\|_{L^4} + \|\nabla^2\mathbb{P}_{\geq k/2}a\|_{L^4})\|f\|_{L^p} \lesssim k^{-1}\|\nabla^2a\|_{L^4}\|f\|_{L^p}. \end{aligned}$$

□

It follows from the definition of w_{q+1} that

$$\int_{\mathbb{T}^3} w_{q+1} dx = \int_{\mathbb{T}^3} \frac{1}{\lambda_{q+1}} \sum_{\bar{\xi} \in \Lambda} \nabla \left(a_{(\bar{\xi})} \eta_{(\bar{\xi})} W_{(\bar{\xi})} \right) dx + \int_{\mathbb{T}^3} \frac{1}{\mu} \sum_{\bar{\xi} \in \Lambda^+} P_{LH} \mathbb{P}_{\neq 0} \left(a_{(\bar{\xi})}^2 \eta_{(\bar{\xi})}^2 \bar{\xi} \right) dx = 0.$$

Hence $\int_{\mathbb{T}^3} \nu(-\Delta)^\theta w_{q+1} dx = 0$ and $\frac{d}{dt} \int_{\mathbb{T}^3} w_{q+1} dx = 0$. We obtain R_{q+1} by plugging $v_{q+1} = v_q + w_{q+1}$ in (2), using (33) and the assumption that (v_q, R_q) solves (2):

$$\begin{aligned} \nabla \cdot R_{q+1} &= \nabla \cdot \left[\mathcal{R}(\nu(-\Delta)^\theta w_{q+1} + \partial_t w_{q+1}^{(p)} + \partial_t w_{q+1}^{(c)}) + v_q \otimes w_{q+1} + w_{q+1} \otimes v_q \right] \\ &\quad + \nabla \cdot \left[(w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)}) \right] \\ &\quad \left[\nabla \cdot (w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} - R_q) + \partial_t w_{q+1}^{(t)} \right] + \nabla(p_{q+1} - p_q) \\ &:= \nabla \cdot (\tilde{R}_{linear} + \tilde{R}_{corrector} + \tilde{R}_{oscillation}) + \nabla(p_{q+1} - p_q). \end{aligned}$$

It follows from Lemma 4 that

$$\begin{aligned} \|\tilde{R}_{corrector}\|_{L_t^\infty L_x^p} &\lesssim \left(\|w_{q+1}^{(c)}\|_{L_t^\infty L_x^{2p}} + \|w_{q+1}^{(t)}\|_{L_t^\infty L_x^{2p}} \right) \left(\|w_{q+1}\|_{L_t^\infty L_x^{2p}} + \|w_{q+1}^{(p)}\|_{L_t^\infty L_x^{2p}} \right) \\ &\lesssim (\sigma r + \mu^{-1} r^{3/2}) r^{3-3/p} \mathcal{C}_1. \end{aligned}$$

Noting that $\nabla \times \frac{w_{q+1}^{(p)}}{\lambda_{q+1}} = w_{q+1}^{(p)} + w_{q+1}^{(c)}$, Lemma 4 and (34) yield that

$$\begin{aligned} & \|\tilde{R}_{linear}\|_{L_t^\infty L_x^p} \\ & \lesssim \lambda_{q+1}^{-1} \|\partial_t \mathcal{R} \nabla \times (w_{q+1}^{(p)})\|_{L_t^\infty L_x^p} + \|\mathcal{R}(\nu(-\Delta)^\theta w_{q+1})\|_{L_t^\infty L_x^p} \\ & \quad + \|v_q \otimes w_{q+1} + w_{q+1} \otimes v_q\|_{L_t^\infty L_x^p} \\ & \lesssim \lambda_{q+1}^{-1} \|\partial_t w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} + \||\nabla|^{2\theta-1} w_{q+1}\|_{L_t^\infty L_x^p} + \|v_q\|_{C^0} \|w_{q+1}\|_{L_t^\infty L_x^p} \\ & \lesssim \sigma \mu r^{5/2-3/p} \mathcal{C}_2 + r^{3/2-3/p} (\lambda_{q+1}^{2\theta-1} + \|v_q\|_{C^0}) \mathcal{C}_3. \end{aligned} \tag{36}$$

This is the crucial estimate to control the fractional viscosity. If we assume that $p \sim 1, r \sim \lambda_{q+1}^{-1}$, we must have $\theta < 5/4$ in order that the second term in (36) is small for λ_{q+1} sufficiently large.

It remains to estimate $\tilde{R}_{oscillation}$, which can be handled in the same way as in [5]. It follows from (19) that

$$\begin{aligned} \nabla \cdot (w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} - R_q) &= \nabla \cdot \left(\sum_{\bar{\xi}, \bar{\xi}' \in \Lambda} a_{(\bar{\xi})} a_{(\bar{\xi}')} \mathbb{W}_{\bar{\xi}} \otimes \mathbb{W}_{(\bar{\xi}')} - R_q \right) \\ &= \nabla \cdot \left(\sum_{\bar{\xi}, \bar{\xi}' \in \Lambda} a_{(\bar{\xi})} a_{(\bar{\xi}')} \mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(\bar{\xi}')} \right) + \nabla \rho \\ &:= \sum_{\bar{\xi}, \bar{\xi}' \in \Lambda} E_{(\bar{\xi}, \bar{\xi}')} + \nabla \rho. \end{aligned}$$

Since $E_{(\bar{\xi}, \bar{\xi}')}$ has zero mean, we can split it as

$$\begin{aligned} E_{(\bar{\xi}, \bar{\xi}')} + E_{(\bar{\xi}', \bar{\xi})} &= \mathbb{P}_{\neq 0} \left(\nabla(a_{(\bar{\xi})} a_{(\bar{\xi}')}) \cdot (\mathbb{P}_{\geq \lambda_{q+1}\sigma/2} (\mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(\bar{\xi}')} + \mathbb{W}_{\bar{\xi}'} \otimes \mathbb{W}_{(\bar{\xi})})) \right) \\ &\quad + \mathbb{P}_{\neq 0} \left(a_{(\bar{\xi})} a_{(\bar{\xi}')} \nabla \cdot (\mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(\bar{\xi}')} + \mathbb{W}_{\bar{\xi}'} \otimes \mathbb{W}_{(\bar{\xi})}) \right) \\ &:= E_{(\bar{\xi}, \bar{\xi}', 1)} + E_{(\bar{\xi}, \bar{\xi}', 2)}. \end{aligned}$$

Using (15), (34) and (35), we obtain

$$\begin{aligned} \|\mathcal{R}E_{(\bar{\xi}, \bar{\xi}', 1)}\|_{L_t^\infty L_x^p} &\lesssim \| |\nabla|^{-1} E_{(\bar{\xi}, \bar{\xi}', 1)} \|_{L_t^\infty L_x^p} \\ &\lesssim (\lambda_{q+1}\sigma)^{-1} \|a_{(\bar{\xi})} a_{(\bar{\xi}')} \|_{C^3} \|\mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(\bar{\xi}')} \|_{L_t^\infty L_x^p} \\ &\lesssim (\lambda_{q+1}\sigma)^{-1} \|a_{(\bar{\xi})} a_{(\bar{\xi}')} \|_{C^3} \|\mathbb{W}_{(\bar{\xi})} \|_{L_t^\infty L_x^{2p}} \|\mathbb{W}_{(\bar{\xi}')} \|_{L_t^\infty L_x^{2p}} \\ &\lesssim (\lambda_{q+1}\sigma)^{-1} r^{3-3/p} \mathcal{C}_3. \end{aligned}$$

Recall the vector identity $A \cdot \nabla B + B \cdot \nabla A = \nabla(A \cdot B) - A \times (\nabla \times B) - B \times (\nabla \times A)$.

For $\bar{\xi}, \bar{\xi}' \in \Lambda$, using the anti-symmetry of the cross product, we can write

$$\begin{aligned} &\nabla \cdot (\mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(\bar{\xi}')} + \mathbb{W}_{(\bar{\xi}')} \otimes \mathbb{W}_{(\bar{\xi})}) \\ &= \left(W_{(\bar{\xi})} \otimes W_{(\bar{\xi}')} + W_{(\bar{\xi}')} \otimes W_{(\bar{\xi})} \right) \nabla \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \right) + \eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \left(W_{(\bar{\xi})} \cdot \nabla W_{(\bar{\xi}')} + W_{(\bar{\xi}')} \cdot \nabla W_{(\bar{\xi})} \right) \\ &= \left(W_{(\bar{\xi}')} \cdot \nabla \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \right) \right) W_{(\bar{\xi})} + \left(W_{(\bar{\xi})} \cdot \nabla \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \right) \right) W_{(\bar{\xi}')} + \eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \nabla \left(W_{(\bar{\xi})} \cdot W_{(\bar{\xi}')} \right). \end{aligned}$$

For the term $E_{(\bar{\xi}, \bar{\xi}', 2)}$, first consider the case $\bar{\xi} + \bar{\xi}' \neq 0$. It follows from the above identity and (14) that

$$\begin{aligned} &a_{(\bar{\xi})} a_{(\bar{\xi}')} \nabla \cdot (\mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(\bar{\xi}')} + \mathbb{W}_{(\bar{\xi}')} \otimes \mathbb{W}_{(\bar{\xi})}) \\ &= a_{(\bar{\xi})} a_{(\bar{\xi}')} \nabla \cdot \mathbb{P}_{\geq \lambda_{q+1}/10} \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \left(W_{(\bar{\xi})} \otimes W_{(\bar{\xi}')} + W_{(\bar{\xi}')} \otimes W_{(\bar{\xi})} \right) \right) \\ &= a_{(\bar{\xi})} a_{(\bar{\xi}')} \mathbb{P}_{\geq \lambda_{q+1}/10} \left(\nabla \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \right) \cdot \left(W_{(\bar{\xi})} \otimes W_{(\bar{\xi}')} + W_{(\bar{\xi}')} \otimes W_{(\bar{\xi})} \right) \right) \\ &\quad + a_{(\bar{\xi})} a_{(\bar{\xi}')} \mathbb{P}_{\geq \lambda_{q+1}/10} \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \nabla \left(W_{(\bar{\xi})} \cdot W_{(\bar{\xi}')} \right) \right) \\ &= a_{(\bar{\xi})} a_{(\bar{\xi}')} \mathbb{P}_{\geq \lambda_{q+1}/10} \left(\nabla \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \right) \cdot \left(W_{(\bar{\xi})} \otimes W_{(\bar{\xi}')} + W_{(\bar{\xi}')} \otimes W_{(\bar{\xi})} \right) \right) \\ &\quad + \nabla \left(a_{(\bar{\xi})} a_{(\bar{\xi}')} \mathbb{W}_{(\bar{\xi})} \cdot \mathbb{W}_{(\bar{\xi}')} \right) - \nabla \left(a_{(\bar{\xi})} a_{(\bar{\xi}')} \right) \mathbb{P}_{\geq \lambda_{q+1}/10} \left(\mathbb{W}_{(\bar{\xi})} \cdot \mathbb{W}_{(\bar{\xi}')} \right) \\ &\quad - a_{(\bar{\xi})} a_{(\bar{\xi}')} \mathbb{P}_{\geq \lambda_{q+1}/10} \left(\left(W_{(\bar{\xi})} \cdot W_{(\bar{\xi}')} \right) \nabla \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \right) \right), \end{aligned}$$

where the second term is a pressure, the third can be estimated analogously to $E_{(\bar{\xi}, \bar{\xi}', 1)}$. Also note that the first and fourth term can be estimated analogously. Using (16), (34) and (35), we obtain

$$\begin{aligned} & \|\mathcal{R} \left(a_{(\bar{\xi})} a_{(\bar{\xi}')} \mathbb{P}_{\geq \lambda_{q+1}/10} \left(\nabla \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \right) \cdot \left(W_{(\bar{\xi})} \otimes W_{(\bar{\xi}')} + W_{(\bar{\xi}')} \otimes W_{(\bar{\xi})} \right) \right) \right) \|_{L_t^\infty L_x^p} \\ & \lesssim \lambda_{q+1}^{-1} \|a_{(\bar{\xi})} a_{(\bar{\xi}')} \|_{C^3} \|\nabla \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \right) \|_{L_t^\infty L_x^p} \\ & \lesssim \sigma r^{4-3/p} \mathcal{C}_3. \end{aligned}$$

Now consider $E_{(\bar{\xi}, -\bar{\xi}, 2)}$. We can write

$$\begin{aligned} \nabla \cdot (\mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(-\bar{\xi})} + \mathbb{W}_{(-\bar{\xi})} \otimes \mathbb{W}_{(\bar{\xi})}) &= \left(W_{(-\bar{\xi})} \cdot \nabla \eta_{(\bar{\xi})}^2 \right) W_{(\bar{\xi})} + \left(W_{(\bar{\xi})} \cdot \nabla \eta_{(\bar{\xi})}^2 \right) W_{(-\bar{\xi})} \\ &= (A_{\bar{\xi}} \cdot \nabla \eta_{(\bar{\xi})}^2) A_{\bar{\xi}} + ((\bar{\xi} \times A_{\bar{\xi}}) \cdot \nabla \eta_{(\bar{\xi})}^2) (\bar{\xi} \times A_{\bar{\xi}}) = \nabla \xi_{(\bar{\xi})}^2 - (\bar{\xi} \cdot \nabla \eta_{(\bar{\xi})}^2) \bar{\xi} = \nabla \eta_{(\bar{\xi})}^2 - \frac{\bar{\xi}}{\mu} \partial_t \eta_{(\bar{\xi})}^2, \end{aligned}$$

where we use (11) and the fact that $\{\bar{\xi}, A_{\bar{\xi}}, \bar{\xi} \times A_{\bar{\xi}}\}$ forms an orthonormal basis of \mathbb{R}^3 . Therefore, we can write

$$\begin{aligned} E_{(\bar{\xi}, -\bar{\xi}, 2)} &= \mathbb{P}_{\neq 0} \left(a_{(\bar{\xi})}^2 \nabla \mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \eta_{(\bar{\xi})}^2 - a_{(\bar{\xi})}^2 \frac{\bar{\xi}}{\mu} \partial_t \eta_{(\bar{\xi})}^2 \right) \\ &= \nabla \left(a_{(\bar{\xi})}^2 \mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \eta_{(\bar{\xi})}^2 \right) - \mathbb{P}_{\neq 0} \left(\mathbb{P}_{\geq \lambda_{q+1}\sigma/2} (\eta_{(\bar{\xi})}^2) \nabla a_{(\bar{\xi})}^2 \right) \\ &\quad - \mu^{-1} \partial_t \mathbb{P}_{\neq 0} \left(a_{(\bar{\xi})}^2 \eta_{(\bar{\xi})}^2 \bar{\xi} \right) + \mu^{-1} \mathbb{P}_{\neq 0} \left(\partial_t \left(a_{(\bar{\xi})}^2 \right) \eta_{(\bar{\xi})}^2 \bar{\xi} \right). \end{aligned}$$

Using the identity $\text{Id} - \mathbb{P}_{LH} = \nabla \Delta^{-1} \text{div}$, we obtain

$$\begin{aligned} \sum_{\bar{\xi}} E_{(\bar{\xi}, -\bar{\xi}, 2)} + \partial_t w_{q+1}^{(t)} &= \nabla \sum_{\bar{\xi}} \left(a_{(\bar{\xi})}^2 \mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \eta_{(\bar{\xi})}^2 \right) - \nabla \sum_{\bar{\xi}} \mu^{-1} \Delta^{-1} \nabla \cdot \partial_t \left(a_{(\bar{\xi})}^2 \eta_{(\bar{\xi})}^2 \bar{\xi} \right) \\ &\quad - \sum_{\bar{\xi}} \mathbb{P}_{\neq 0} \left(\mathbb{P}_{\geq \lambda_{q+1}\sigma/2} (\eta_{(\bar{\xi})}^2) \nabla a_{(\bar{\xi})}^2 \right) + \mu^{-1} \sum_{\bar{\xi}} \mathbb{P}_{\neq 0} \left(\partial_t \left(a_{(\bar{\xi})}^2 \right) \eta_{(\bar{\xi})}^2 \bar{\xi} \right), \end{aligned}$$

where the first and second terms are pressure terms. Using (16), (34) and (35), we obtain

$$\begin{aligned} \|\mathcal{R} \mathbb{P}_{\neq 0} \left(\mathbb{P}_{\geq \lambda_{q+1}\sigma/2} (\eta_{(\bar{\xi})}^2) \nabla a_{(\bar{\xi})}^2 \right) \|_{L_t^\infty L_x^p} &\lesssim (\lambda_{q+1}\sigma)^{-1} \|\eta_{(\bar{\xi})}\|_{L_t^\infty L_x^{2p}}^2 \mathcal{C}_3 \\ &\lesssim (\lambda_{q+1}\sigma)^{-1} r^{3-3/p} \mathcal{C}_3. \end{aligned}$$

It follows from (16) and (34) that

$$\begin{aligned} \mu^{-1} \|\mathcal{R} \mathbb{P}_{\neq 0} \left(\partial_t \left(a_{(\bar{\xi})}^2 \right) \eta_{(\bar{\xi})}^2 \bar{\xi} \right) \|_{L_t^\infty L_x^p} &\lesssim \mu^{-1} \|\partial_t \left(a_{(\bar{\xi})}^2 \right) \eta_{(\bar{\xi})}^2 \bar{\xi}\|_{L_t^\infty L_x^p} \\ &\lesssim \mu^{-1} r^{3-3/p} \mathcal{C}_1. \end{aligned}$$

Let us now give the explicit definition of $\tilde{R}_{oscillation}$:

$$\begin{aligned} \tilde{R}_{oscillation} = & \sum_{\bar{\xi}, \bar{\xi}' \in \Lambda} \mathbb{P}_{\neq 0} \left(\nabla(a_{(\bar{\xi})} a_{(\bar{\xi}')}) \cdot (\mathbb{P}_{\geq \lambda_{q+1}\sigma/2} (\mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(\bar{\xi}')} + \mathbb{W}_{\bar{\xi}'} \otimes \mathbb{W}_{(\bar{\xi})})) \right) \\ & + \sum_{\bar{\xi}, \bar{\xi}' \in \Lambda, \bar{\xi} \neq \bar{\xi}'} a_{(\bar{\xi})} a_{(\bar{\xi}')} \mathbb{P}_{\geq \lambda_{q+1}/10} \left(\nabla \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \right) \cdot \left(W_{(\bar{\xi})} \otimes W_{(\bar{\xi}')} + W_{(\bar{\xi}')} \otimes W_{(\bar{\xi})} \right) \right) \\ & - \sum_{\bar{\xi}, \bar{\xi}' \in \Lambda, \bar{\xi} \neq \bar{\xi}'} \nabla \left(a_{(\bar{\xi})} a_{(\bar{\xi}')} \right) \mathbb{P}_{\geq \lambda_{q+1}/10} \left(\mathbb{W}_{(\bar{\xi})} \cdot \mathbb{W}_{(\bar{\xi}')} \right) \\ & - \sum_{\bar{\xi}, \bar{\xi}' \in \Lambda, \bar{\xi} \neq \bar{\xi}'} a_{(\bar{\xi})} a_{(\bar{\xi}')} \mathbb{P}_{\geq \lambda_{q+1}/10} \left(\left(W_{(\bar{\xi})} \cdot W_{(\bar{\xi}')} \right) \nabla \left(\eta_{(\bar{\xi})} \eta_{(\bar{\xi}')} \right) \right) \\ & - \sum_{\bar{\xi} \in \Lambda} \mathbb{P}_{\neq 0} \left(\mathbb{P}_{\geq \lambda_{q+1}\sigma/2} (\eta_{(\bar{\xi})}^2) \nabla a_{(\bar{\xi})}^2 \right) + \mu^{-1} \sum_{\bar{\xi} \in \Lambda} \mathbb{P}_{\neq 0} \left(\partial_t \left(a_{(\bar{\xi})}^2 \right) \eta_{(\bar{\xi})}^2 \bar{\xi} \right). \end{aligned}$$

Finally, we estimate the time support of R_{q+1} . Using (25) we obtain

$$\text{supp}_t R_{q+1} \subset \text{supp}_t w_{q+1} \cup \text{supp}_t R_q \subset N_{\delta_{q+1}}(\text{supp}_t R_q).$$

Now we choose the parameters r, σ, μ . Fix α so that

$$\max\{0, \frac{2}{3}(2\theta - 1)\} < \alpha < 1,$$

which is possible since $\theta \in (-\infty, 5/4)$. Fix

$$r = \lambda_{q+1}^\alpha, \quad \sigma = \lambda_{q+1}^{-(\alpha+1)/2}, \quad \mu = \lambda_{q+1}^{(5\alpha+1)/4}. \quad (37)$$

Clearly (27) is satisfied. Choose $p > 1$ sufficiently close to 1 so that

$$\begin{aligned} -\frac{\alpha+1}{2} + \frac{5\alpha+1}{4} + \left(\frac{5}{2} - \frac{3}{p}\right)\alpha &< 0, \quad \left(\frac{3}{2} - \frac{3}{p}\right)\alpha + \max(0, 2\theta - 1) < 0, \\ -\frac{5\alpha+1}{4} + \left(\frac{9}{2} - \frac{3}{p}\right)\alpha &< 0, \quad -\frac{1-\alpha}{2} + \left(3 - \frac{3}{p}\right)\alpha < 0. \end{aligned}$$

Note that \mathcal{C}_N is independent of λ_{q+1} , due to (24). Combining the above estimates with Lemma 4, it is easy to check that, by taking λ_{q+1} sufficiently large, we arrive at (4), (6) and (7). This completes the proof of Lemma 1.

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